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Playing with Numbers, with Fermions and Bosons

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Abstract We construct nonlinear maps which realize the fermionization of bosons and the bosonization of fermions with the view of obtaining states coding naturally integers in standard or in binary basis. Specifically, with reference to spin $\frac{1}{2}$ systems, we derive raising and lowering bosonic operators in terms of standard fermionic operators and vice versa. The crucial role of multiboson operators in the whole construction is emphasized.

Keywords Fermionization · Bosonization · Multiboson algebra

1 Introduction

Quantum computation implies a deep one-to-one correspondence between the microscopic states of matter, e.g., spin systems, many-fermion and many-boson systems, harmonic oscillators, and the numbers of a typically finite ring of integers, e.g., $\mathbb{Z}_2^{\otimes q}$.

According to this view, the process of quantum computation, which entails the manipulation of physical quantum states, is basically equivalent to manipulate numbers, at least as long as one considers pure states only.

On the one hand, an appropriate basis, 2, 10, ..., is required to perform calculations. On the other hand, the unitary transformations corresponding to logical operations (quantum gates) are clearly applied on physical quantum systems. And one knows that, disregarding for the moment the special case of anyons, such systems, as any of the 'objects' of quantum mechanics, cannot but be either fermions or bosons.

Dedicated to Giuseppe Castagnoli for his 65th birthday.

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Here we play with Fock space, with creation and annihilation operators for both bosons and fermions, to show that it is possible, resorting to nonlinear maps whose complex structure is dictated by the noncommutative algebraic setting underlying the physical description, to establish a reciprocal connection between a representation in terms of fermions and one with bosons, i.e., between the world of the constituents of matter and the world of the messengers of the interaction. We plan to discuss the real physics behind this elsewhere.

2 Fermionization of Bosonic Systems

2.1 Finite Register

Let us first consider finite numbers, the realistic case for both classical and quantum computation, namely the set of integers $\mathbb{N}_q \doteq \{n \mid 0 \le n \le n_{\max}\}$, where, with no loss of generality, we set $n_{\max} = 2^q - 1$. In binary notation

$$n = \sum_{\ell=0}^{q-1} \nu_\ell 2^\ell, \quad \nu_\ell \in \mathbb{Z}_2,\ q \ge 1,\ 0 \le n \le n_{\max}. \tag{1}$$

In quantum terms this means that one is dealing with a finite integer n, that is interpreted either as the eigenvalue of the bosonic occupation number operator $\hat{n}$ or, upon identifying ν_ℓ as the $(\ell+1)$-th binary digit of n, with the collection of eigenvalues of the fermionic occupation number operators $\hat{\nu_\ell}$. Analogously, we associate to n the number-state $|n\rangle$ in Fock space, or, in binary notation, the multifermion state $|\nu_0\rangle \otimes |\nu_1\rangle \otimes \cdots \otimes |\nu_{q-1}\rangle$.

The appropriate space for $|n\rangle$, $\mathfrak{F}_q \doteq \mathrm{span}\{|n\rangle \mid \hat{n}|n\rangle = n|n\rangle;\ 0 \le n \le n_{\max}\}$ is a subspace of the Fock space $\mathfrak{F} = \mathrm{span}\{|n\rangle \mid n \in \mathbb{N}\}$. On the other hand, $|\nu_\ell\rangle$ lives in the 2×2 subspace $\mathfrak{H}_\ell \sim \mathbb{C}^2$ of $\mathfrak{F}_q$. The structure of the fermionic state space is therefore $\mathfrak{H} \doteq (\mathbb{C}^2)^{\otimes q}$. Of course, $\mathfrak{H} \sim \mathfrak{F}_q$:

$$\mathfrak{F}_q \equiv \bigotimes_{\ell=0}^{q-1} \mathfrak{H}_\ell. \tag{2}$$

We define the following fermionized annihilation operators

$$F_m = \sum_{k=m}^{q-2} e^{i\hat{\Phi}_{m,k}} \prod_{j=m}^{k} f_j^\dagger f_{k+1} + e^{i\hat{\Phi}_{m,1-m}} f_m \quad \text{for } 0 \le m \le q-2, \tag{3}$$

$$F_{q-1} = e^{i\hat{\Phi}_{q-1,2-q}} f_{q-1}, \tag{4}$$

where $f_j, f_j^\dagger$ are fermion operators, $\{f_j, f_k^\dagger\} = \delta_{jk}\mathbb{I}$, $\{f_j, f_k\} = 0$, acting on the subspace $\mathfrak{H}_j$. They are endowed with the tensor-product structure

$$f_j = \underbrace{\sigma_z^{(0)} \otimes \cdots \otimes \sigma_z^{(j-1)}}_{j \text{ factors}} \otimes \underbrace{\sigma_+^{(j)}}_{(j+1)\text{-th}} \otimes \underbrace{\mathbb{I}_2^{(j+1)} \otimes \cdots \otimes \mathbb{I}_2^{(q-1)}}_{(q-j-1) \text{ factors}}. \tag{5}$$

In (5) $\sigma_\alpha^{(j)}$, $\alpha = \pm, z$, $j = 0, 1, \ldots, q-1$, $(\sigma_-^{(j)})^\dagger = \sigma_+^{(j)}$ are the Pauli matrices acting in $\mathfrak{H}_j$, $\sigma_z^{(j)}|\nu_j\rangle = e^{i\pi\nu_j}|\nu_j\rangle$, $\sigma_+^{(j)}|\nu_j\rangle = \nu_j|\nu_j - 1\rangle$, $\sigma_-^{(j)}|\nu_j\rangle = (1-\nu_j)|\nu_j + 1\rangle$. Unlike the Lie algebra convention, $\sigma_+^{(j)}$ and $\sigma_-^{(j)}$ play the role of lowering and raising operators, respectively, while $\sigma_z^{(j)}$ flips the phase of $|\nu_j\rangle$ when $\nu_j = 1$.

Notice that $f_j^\dagger f_j = \hat{\nu}_j$, the fermionic number operator, since $(\sigma_-\sigma_+)^{(j)}|\nu_j\rangle = \frac{1}{2}(\mathbb{I}_2 - \sigma_z)^{(j)}|\nu_j\rangle = \nu_j|\nu_j\rangle$, being $\sigma_z^{(j)} = e^{i\pi\hat{\nu}_j} = \mathbb{I}_2 - 2\hat{\nu}_j$.

For $0 \le m \le q-1$, the phase factor operators $\exp(i\hat{\Phi}_{m,k})$ where

$$\hat{\Phi}_{m,k} \equiv \hat{\Phi}_{m,k}(\hat{n}) = \pi(k+m)\left(\sum_{\ell=0}^{m}\hat{\nu}_\ell - \hat{\nu}_m\right), \tag{6}$$

are introduced in (3, 4) to compensate for the changes of sign induced by the action of the matrices σ_z on $|n\rangle$; in particular, $1-m$ and $2-q$ are the 'fictitious' values of k which give $k+m=1$ for the phase factors multiplying a single annihilation or creation fermion operator.

The resulting action of F_m, $F_m^\dagger$ on $|n\rangle$ proves then to be

$$F_m|n\rangle = |n-2^m\rangle, \qquad F_m^\dagger|n\rangle = |n+2^m\rangle. \tag{7}$$

F_m and $F_m^\dagger$ are unitary for $2^m \le n \le 2^q - 1 - 2^m$, they are not otherwise. For example, defining the 'vacuum' state of the register as $|v\rangle = \sum_{i=0}^{2^m-1}\alpha_i|i\rangle$, $\alpha_i \in \mathbb{C}$, $\sum_i|\alpha_i|^2 = 1$, gives $F_m^\dagger F_m|v\rangle = 0 \neq F_m F_m^\dagger|v\rangle = |v\rangle$.

2.2 Infinite Register

If the full ring of integers $\mathbb{N}$ is considered, $n = \sum_{\ell=0}^{\infty}\nu_\ell 2^\ell$, one defines, for $m \ge 0$

$$F_m = \sum_{k=m}^{\infty} e^{i\hat{\Phi}_{m,k}}\prod_{j=m}^{k} f_j^\dagger f_{k+1} + e^{i\hat{\Phi}_{m,1-m}} f_m. \tag{8}$$

F_m and $F_m^\dagger$ are now unitary for $n \ge 2^m$.

2.3 Fermionized and Multiboson Operators

Multiboson (or k-boson) algebras are the generalization of the $h(1)$ algebra $\{\mathbb{I}, a, a^\dagger, \hat{n} \doteq a^\dagger a\}$ [1, 2] In the field of quantum computation, the authors have developed a theoretical approach based on such algebras, paying attention to the construction of both quantum algorithms [3] and, more recently, quantum logical states (codewords) and operators (gates) [4, 5]. Here, we recall that the operators of the k-boson algebra $\{\mathbb{I}, A_k, A_k^\dagger, \hat{N}_k \doteq A_k^\dagger A_k\}$, whose relevant commutation relations are $[\mathbb{I}, \bullet] = 0$, $[A_k, A_k^\dagger] = \mathbb{I}$, $[\hat{N}_k, A_k] = -A_k$, $[\hat{N}_k, A_k^\dagger] = A_k^\dagger$, can be realized in terms of a, $a^\dagger$, $\hat{n}$

$$A_k = a^k C_k(\hat{n}), \quad C_k(\hat{n}) = \sqrt{\left[\!\!\left[\frac{\hat{n}}{k}\right]\!\!\right]\frac{(\hat{n}-k)!}{\hat{n}!}}. \tag{9}$$

Setting $n = sk + t$, with $s \doteq [\![\frac{n}{k}]\!]$ and $t \doteq \{\frac{n}{k}\}$ the residue of n mod k, $0 \le t \le k-1$, the action on the Fock states $|n\rangle = |sk+t\rangle$ proves to be

$$A_k|n\rangle = \sqrt{s}\,|n-k\rangle, \qquad A_k^\dagger|n\rangle = \sqrt{s+1}\,|n+k\rangle, \qquad \hat{N}_k|n\rangle = s\,|n\rangle. \tag{10}$$

Notice also the operator $\hat{D}_k \doteq \hat{n} - k[\![\frac{\hat{n}}{k}]\!]$, whose action is $\hat{D}_k|n\rangle = t|n\rangle$.

The following relation between the fermionized operators and the multiboson operators of the 2^r-boson algebra ($r \geq 0$) holds

$$F_r = A_{2^r}\,\hat{N}_{2^r}^{-\frac{1}{2}} = (\hat{N}_{2^r}+1)^{-\frac{1}{2}}A_{2^r}, \qquad F_r^\dagger = \hat{N}_{2^r}^{-\frac{1}{2}}A_{2^r}^\dagger = A_{2^r}^\dagger(\hat{N}_{2^r}+1)^{-\frac{1}{2}}, \tag{11}$$

showing that, as expected, $F_r F_r^\dagger = \mathbb{I}$ whereas $F_r^\dagger F_r = 0$ on the vacuum and $\mathbb{I}$ elsewhere in the unitarity region. One should indeed be careful as the validity of the above correspondence between fermionized and purely bosonic operators is restricted to the unitarity regions of F_r, $F_r^\dagger$, $2^r \leq n \leq 2^q - 1 - 2^r$ and $n \geq 2^r$, for the finite and the infinite register, respectively.

2.4 Fermionized Operators and Quantum Phase Operators

In the unitarity range $n \geq 1$ of the infinite register, the operator $F_0 = a\hat{n}^{-\frac{1}{2}}$, from (11), can be identified as the fermionic realization of the quantum phase operator $\hat{e}$, $a = \hat{e}\sqrt{\hat{n}}$, i.e., $\hat{e} \equiv F_0$, with $F_0^\dagger = F_0^{-1}$: for the phase-modulus decomposition of the bosonic annihilation operator and the strictly related problem of the number-phase uncertainty, here we cite [6], for obvious reasons, and [7] for a recent review on both topics. Since F_0 is not unitary on the whole range $n \geq 0$, one retrieves the well known results $\hat{e}\hat{e}^\dagger = F_0 F_0^\dagger \equiv \mathbb{I}$, $\hat{e}^\dagger\hat{e} = F_0^\dagger F_0 \equiv \mathbb{I} - |0\rangle\langle 0|$, namely $[F_0, F_0^\dagger] = |0\rangle\langle 0|$, where $|0\rangle$ is the vacuum vector of the register.

For the finite register, the range of values over which F_0 plays the role of unitary quantum phase operator is $1 \leq n \leq 2^q - 2$.

2.5 General Shift of the Number-States

Here, the shift we are considering refers to the transformation $|n\rangle \to |n - z\rangle$, for general values of z in the range $0 \leq z \leq n_{\max}$, being $z = \sum_{\ell=0}^{q-1}\zeta_\ell 2^\ell$, $\zeta_\ell \in \mathbb{Z}_2$. The fermionized annihilation operator $\mathcal{O}(z)$ designed to perform such action, i.e., $\mathcal{O}(z)|n\rangle = |n-z\rangle$, is

$$\mathcal{O}(z) = \prod_{l=0}^{q-1} G_\ell, \qquad G_\ell = \zeta_\ell F_\ell + (1-\zeta_\ell)\mathbb{I}_2^{\otimes q}. \tag{12}$$

3 Bosonization of Fermionic Systems

For infinite register one can conveniently refer to [8], where, in our notation, the following expression for the fermionic operators in terms of bosonic operators is given

$$f_\ell = \left[\frac{\hat{n}!}{(\hat{n}+2^\ell)!}\right]^{\frac{1}{2}} \sum_{m=0}^{\infty}\sum_{r_\ell=0}^{2^\ell-1} (-)^{\mathfrak{p}(r_\ell)} |2^{\ell+1}m + r_\ell\rangle\langle 2^{\ell+1}m + r_\ell| a^{2^\ell}, \tag{13}$$

with $\mathfrak{p}(r_\ell)$ the number of odd binary digits of the integer r_ℓ, that we shall talk about further on.

3.1 Finite Register

In order to write the matrix representation of the fermion operators f_ℓ, $f_\ell^\dagger$ in $\mathfrak{F}_q$, one starts noticing first that the $2^\ell \times 2^\ell$ matrix $\sigma_z^{(0)} \otimes \cdots \otimes \sigma_z^{(\ell-1)}$ is diagonal in $\mathfrak{F}_q$, since of course

$$\left[\sigma_z^{(0)} \otimes \cdots \otimes \sigma_z^{(\ell-1)}\right]_{rs} = (-)^{\mathfrak{p}(r)}\delta_{rs}, \quad 0 \leq r, s \leq 2^\ell - 1. \tag{14}$$

With $0 \le \ell \le q-1$, one obtains therefore the two equivalent expressions

$$
\begin{aligned}
f_\ell &= \sum_{m=0}^{2^{q-\ell-1}-1} \sum_{r_\ell=0}^{2^\ell-1} (-)^{\hat{\mathfrak{p}}} |2^{\ell+1}m + r_\ell\rangle\langle 2^{\ell+1}m + r_\ell + 2^\ell| \\
&= \left[\frac{\hat{n}!}{(\hat{n}+2^\ell)!}\right]^{\frac{1}{2}} \sum_{m=0}^{2^{q-\ell-1}-1} \sum_{r_\ell=0}^{2^\ell-1} (-)^{\hat{\mathfrak{p}}} |2^{\ell+1}m + r_\ell\rangle\langle 2^{\ell+1}m + r_\ell| a^{2^\ell}.
\end{aligned} \tag{15}
$$

The replacement of the bra $\langle 2^{\ell+1}m + r_\ell + 2^\ell|$ with $\langle 2^{\ell+1}m + r_\ell|$ is compensated by the operators a^{2^ℓ} on the right and $[\frac{\hat{n}!}{(\hat{n}+2^\ell)!}]^{\frac{1}{2}}$ on the left, while the sum of the projectors appearing in the above equation for f_ℓ can be expressed in terms of boson operators as

$$
\sum_{m=0}^{2^{q-\ell-1}-1} |2^{\ell+1}m + r_\ell\rangle\langle 2^{\ell+1}m + r_\ell| = \prod_{k=0}^{\ell} \cos^2\left(\frac{\pi(\hat{n}-r_\ell)}{2^{k+1}}\right) \doteq P_\ell(\hat{n}). \tag{16}
$$

In (15) we have introduced the operator $(-)^{\hat{\mathfrak{p}}}$, function of $\hat{n}$, whose expression in terms of bosonic operators needs to be explicitly derived for the consistency of the whole construction.

As concerns this latter point, one notices that $\mathfrak{p}(r_\ell)$ in (13) can be regarded as the eigenvalue of the operator $\sum_{k=0}^{q-1} \hat{\nu}_k$ when it acts on the number-state $|r_\ell\rangle$. Since $(-)^{\hat{\mathfrak{p}}}$ operates on ket $|N_\ell\rangle \doteq |2^{\ell+1}m + r_\ell\rangle$, in order to get the required eigenvalue $(-)^{\mathfrak{p}(r_\ell)}$ one needs to perform an intermediate step. Notice first that, in the framework of multiboson algebras, the 'residue' operator

$$
\hat{D}_{2^\ell} \doteq \hat{n} - 2^\ell \left[\!\left[\frac{\hat{n}}{2^\ell}\right]\!\right], \tag{17}
$$

can be defined. Its action on the relevant number-state $|N_\ell\rangle$ is readily evaluated as $[\![\frac{N_\ell}{2^\ell}]\!] = 2m$, giving $\hat{D}_{2^\ell}|N_\ell\rangle = r_\ell|N_\ell\rangle$. This means that indeed $\hat{D}_{2^\ell}$ extracts from $|N_\ell\rangle$ the integer r_ℓ we are interested in. To make the construction explicit one then evaluates $f_\ell^\dagger f_\ell$, which leads to defining the necessary projectors $Q_\ell(\hat{n})$:

$$
\begin{aligned}
f_\ell^\dagger f_\ell &= \sum_{m=0}^{2^{q-\ell-1}-1} \sum_{r_\ell=0}^{2^\ell-1} (-)^{\hat{\mathfrak{p}}} |2^{\ell+1}m + r_\ell + 2^\ell\rangle\langle 2^{\ell+1}m + r_\ell + 2^\ell| \\
&= \sum_{r_\ell=0}^{2^\ell-1} \prod_{k=0}^{\ell} \cos^2\left(\frac{\pi(\hat{n}-r_\ell-2^\ell)}{2^{k+1}}\right) \doteq Q_\ell(\hat{n}) \equiv \hat{\nu}_\ell,
\end{aligned} \tag{18}
$$

namely $Q_\ell(\hat{n})|n\rangle = \nu_\ell|n\rangle$. It is straightforward to infer, now, that for r_ℓ expressed in binary form as $r_\ell = \sum_{s=0}^{\ell-1} \mu_s 2^s$, $\forall \ell \ge 1$, $\mu_s \in \mathbb{Z}_2$, $r_0 = 0$, one has $Q_s(\hat{D}_{2^\ell})|n\rangle = \mu_s|n\rangle$. Here, of course

$$
Q_s(\hat{D}_{2^\ell}) = \sum_{r_\ell=0}^{2^\ell-1} \prod_{k=0}^{s} \cos^2\left(\frac{\pi(\hat{D}_{2^\ell}-r_\ell-2^\ell)}{2^{k+1}}\right). \tag{19}
$$

We have thus, formally,

$$\hat{\mathfrak{p}} = \sum_{s=0}^{\ell-1} Q_s(\hat{D}_{2^\ell}), \qquad \mathfrak{p}(r_\ell) = \sum_{s=0}^{\ell-1} \mu_s, \tag{20}$$

implying $(-)^{\hat{\mathfrak{p}}}|N_\ell\rangle = (-)^{\mathfrak{p}(r_\ell)}|N_\ell\rangle$. Finally, upon utilizing (16), (20) and $\exp[i\pi\sum_{s=0}^{\ell-1} Q_s(\hat{D}_{2^\ell})] = \prod_{s=0}^{\ell-1}(\mathbb{I}-2Q_s(\hat{D}_{2^\ell}))$, one accomplishes the full bosonization of f_ℓ

$$f_\ell = \left[\frac{\hat{n}!}{(\hat{n}+2^\ell)!}\right]^{\frac{1}{2}} \sum_{r_\ell=0}^{2^\ell-1} \prod_{s=0}^{\ell-1} (\mathbb{I} - 2Q_s(\hat{D}_{2^\ell}))\, P_\ell(\hat{n})\, a^{2^\ell}. \tag{21}$$

4 Conclusions and Perspectives

We have constructed operators whose action on number-states can be summarized as follows. The fermionized operators F_m and $F_m^\dagger$ act on $|n\rangle$, with n expressed in binary form, giving the states $|n-2^m\rangle$ and $|n+2^m\rangle$, respectively. It is worth emphasizing that while the shifted states can be obtained resorting to the single and multiboson operators, a^{2^m}, $(a^{(\dagger)})^{2^m}$ and A_{2^m}, $A_{2^m}^\dagger$, the action of F_m and $F_m^\dagger$ looks comparatively simpler and more attractive. On the other hand, the operators F_m, $F_m^\dagger$, unlike their single and multiboson counterparts, do not close a $h(1)$ algebra. In the unitarity region, for instance, $[F_m, F_m^\dagger] = 0$; then one may be led to define the action of these new operators as a sort of abelianization of the $h(1)$ algebra. Notice also that, for $m = 0$, a close relation with the largely investigated problem of the number-phase uncertainty [9] has emerged.

As for the bosonized operators f_ℓ and $f_\ell^\dagger$, they are so designed as to act on the state $|n\rangle \in \mathfrak{F}_q$, with n now expressed in standard notation, giving $f_\ell|n\rangle = \nu_\ell|n-2^\ell\rangle$ and $f_\ell^\dagger|n\rangle = (1-\nu_\ell)|n+2^\ell\rangle$, respectively. Here the fundamental anticommutation relations are preserved and the occupation number operator $\hat{\nu}_\ell$ is readily retrieved as $f_\ell^\dagger f_\ell|n\rangle = \nu_\ell|n\rangle$.

We also mention that, while in the present paper only a binary basis, corresponding to spin $\frac{1}{2}$ physical systems, is considered, work is in progress to extend the construction to general basis b. Setting $b = 2j+1$, the latter is quite naturally related to the spin j representations of $su(2)$, with j integer or half odd.

Finally, we conjecture that also the question of the prospective relation between this work and the representation of Lie superalgebras may possibly be put forward for consideration.

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